GROUP ACTIONS ON GRAPHS RELATED TO KRISHNAN-SUNDER SUBFACTORS

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ABSTRACT. We describe the principal graphs of the subfactors studied by Krishnan and Sunder in terms of group actions on Cayley-type graphs. This leads to the construction of a tower of *tree algebras*, for every positive integer k, which are symmetries of the Krishnan-Sunder subfactors of index k^2. Using our theory, we prove that the principal graph of the irreducible infinite depth subfactor of index 9 constructed by Krishnan and Sunder is not a tree, contrary to their expectations. We also show that the principal graphs of the Krishnan-Sunder subfactors of index 4 are the affine A and D Coxeter graphs.

1. BACKGROUND AND INTRODUCTION

Given a *symmetric commuting square* of finite dimensional C^*-algebras

$$\mathcal{C}: \quad \begin{array}{ccc} B & \subset & D \\ \cup & & \cup \\ A & \subset & C \end{array}$$

there is a well-known way to construct a subfactor $R_\mathcal{C} \subset R$ of the hyperfinite II_1 factor from $\mathcal{C}$ (for commuting squares see [14], [4], [9]; for the construction see [19], [9]). This construction is quite general. A consequence of Popa's work is that any *finite depth* subfactor of the hyperfinite II_1 subfactor can be constructed in this way ([15], [16], [13]). However, the construction still keeps secrets. Ocneanu compactness ([12], [9]) provides a method of computing the *standard invariant* of $R_\mathcal{C} \subset R$ and, in particular, the *principal graph* ([8], [4]). (The principal graph Γ is a possibly infinite graph with a distinguished root vertex $*$ and an eigenvector τ with eigenvalue the Jones index $[R : R_\mathcal{C}]$. It is a combinatoric encoding of the tower of higher relative commutants $(R'_\mathcal{C} \cap R_n)_n$ where $R_\mathcal{C} \subset R_0 = R \subset R_1 \subset R_2 \ldots$ is the Jones tower of $R_\mathcal{C} \subset R$.) Although in theory we can compute the nth higher relative commutant by solving a finite dimensional linear algebra problem, simply writing down the necessary equations takes time exponential in n, and the computation is in general intractable.

In [10], U. Krishnan and V. Sunder consider, in their words, "a seemingly simple class" of commuting squares. Let k and p be positive integers and U a unitary matrix in $M_p(\mathbb{C}) \otimes M_k(\mathbb{C})$. A vertex model commuting square ([9]) is a commuting

square $\mathcal{C}$ of the form

$$(1.1) \qquad\qquad U(1 \otimes M_k(\mathbb{C}))U^* \quad \subset \quad M_p(\mathbb{C}) \otimes M_k(\mathbb{C})$$

$$\cup \qquad\qquad\qquad \cup$$

$$\mathbb{C} \qquad\quad \subset \quad M_p(\mathbb{C}) \otimes 1$$

Following Krishnan and Sunder's notation, we denote $R_{\mathcal{C}} \subset R$ by $R_U \subset R$ instead. The Jones index of $R_U \subset R$ is k^2. A Krishnan-Sunder subfactor is a subfactor $R_U \subset R$ for which U is a permutation matrix. Krishnan and Sunder compute the principal graphs of all such subfactors of finite depth (i.e. Γ is finite) in the case $k = p = 2$ or $k = p = 3$. In their analysis of Krishnan-Sunder subfactors they construct a discrete group G and show that the vertices of Γ correspond to finite dimensional representations of certain subgroups of G.

We add a new twist to Krishnan and Sunder's analysis of $R_U \subset R$ by constructing a graph $\mathcal{H}$ and a faithful action of G on $\mathcal{H}$. The graph $\mathcal{H}$ has the property that each vertex of $\mathcal{H}$ is adjacent to exactly k edges; also, $\mathcal{H}$ is infinite if and only if $R_U \subset R$ has infinite depth. We formulate Γ in terms of $\mathcal{H}$ and the G-action as follows. Let $(P_n)_n$ be the tower of path algebras on $\mathcal{H}$ with the trace given by the constant weight vector on $\mathcal{H}$. Consider the subtower $(B_n)_n$ that commutes with the action of G on $\mathcal{H}$. Then $(B_n)_n$ inherits Jones projections and a k-Markov trace from the path algebras. We show that there is a trace-preserving $*$-isomorphism of $(B_n)_n$ with the tower of higher relative commutants of $R_U \subset R$ that preserves the Jones projections. From this point it is straightforward to describe Γ in terms of the representation theory of subgroups of G. As an example, we apply our theory to the case $k = 2$ and show that the principal graphs of the Krishnan-Sunder subfactors of index 4 are $A_{2n-1}^{(1)}$, $n \geq 1$, and $D_n^{(1)}$, $n \geq 4$. Our theory also yields a short proof of Theorem 37 in [10], which describes the principal graph of a particular infinite depth Krishnan-Sunder subfactor.

We remark that if H and $\mathcal{G}$ are the group and graph of the index p^2 Krishnan-Sunder subfactor arising from (1.1), then $\mathcal{G}$ and $\mathcal{H}$ are bipartite generalizations of the Cayley graphs of G and H, respectively (see Example 6.2.2 of [9] for an example of this type of Cayley graph).

In [10], Krishnan and Sunder state that they believe the principal graphs of their two infinite depth subfactors are trees. We show that this is not the case, and in particular that the principal graph of the irreducible infinite depth subfactor is not a tree. We will show in a future note that the principal graph of the second infinite depth subfactor is not a tree either.

An outcome of our graphical formulation is the existence, for each k, of a tower of *tree algebras* $(T_n(k))_n$ that are symmetries of all Krishnan-Sunder subfactors of index k^2. By this we mean they are contained in a canonical way in the tower of higher relative commutants of any Krishnan-Sunder subfactor of index k^2. We defined the tree algebras previously in [19] and showed in Corollary 6.2.5 and Corollary 6.2.6 of [19] that they are in fact a Popa system (λ-lattice in [17]) and a sub-Popa system (or sub-λ-lattice) of the standard invariant of any Krishnan-Sunder subfactor of index k^2. Hence, the tree algebras $(T_n(k))_n$ are themselves the higher relative commutants of a II_1 subfactor of index k^2, which is irreducible, self-dual, and of infinite depth (Proposition 6.3.2 and Theorem 3.4.3 of [19]). However, the tree

algebras are not the higher relative commutants of a Krishnan-Sunder subfactor (see Proposition 9.6).

This paper is organized into 11 sections. Sections 2 and 4 contain background and notation. General theory is developed in Section 3 and Sections 5 through 9. Sections 10 and 11 contain examples and calculations. The reader is encouraged to read the examples in parallel with the rest of the paper.

ACKNOWLEDGEMENTS

2. NOTATION AND JONES DIAGRAMS

Let V^n, $n = 0, 1, 2, \ldots$, be the complex vector space with orthonormal basis consisting of words of length n on the alphabet $[k]$ ($V^0 = \mathbb{C}e$, where e is the empty word). We may identify V^n with $(V^1)^{\otimes n}$ via the natural bijection of bases.

Then

$$\mathbb{C} \cong \mathrm{End}(V^0) \subset \mathrm{End}(V^2) \subset \cdots \subset \mathrm{End}(V^n) \subset \ldots$$

is a sequence of $*$-algebras with the inclusion map $F \mapsto F \otimes 1$. If tr_n is the unique trace on $\mathrm{End}(V^n)$ normalized so that $\mathrm{tr}_n(1) = 1$, then tr_n respects the inclusion maps and extends to a trace tr on $\bigcup_n \mathrm{End}(V^n)$.

We briefly discuss the conventions of Jones diagrams, a convenient way to work with Ocneanu compactness [9]. Given $F \in \mathrm{End}(V^n)$, view F as a matrix with respect to the basis $[k]^n$. Given a word $w \in [k]^n$, denote by w^t the word w spelled backwards. Denote the entry of F in row $w_b = b_1 b_2 \ldots b_n \in [k]^n$ and column $w_a = a_1 a_2 \ldots a_n \in [k]^n$ by F_w, where $w = w_b w_a^t = b_1 b_2 \ldots b_n a_n \ldots a_2 a_1$. We also represent this entry by a box surrounded by $2n$ oriented strings labeled as below:

Let U be a unitary matrix in $M_p(\mathbb{C}) \otimes M_k(\mathbb{C})$. Index the rows and columns of U by the set $[p] \times [k]$, where $[p]$ is a set of p elements and $[k]$ is a set of k elements. Following Krishnan and Sunder's notation, we denote elements of $[p]$

by Greek letters and elements of $[k]$ by Roman letters. Denote the entry of U in column (α, a) and row (β, b) by $U_{\alpha a}^{\beta b}$. The square in (1.1) is a commuting square if and only if U satisfies Ocneanu's *biunitary condition* ([13], [7]). In the vertex model case the biunitary condition on U is that both U and its "block transpose conjugate" $\widetilde{U} \in M_p(\mathbb{C}) \otimes M_k(\mathbb{C})$, defined by

$$\widetilde{U}_{\beta a}^{\alpha b} = \overline{U_{\alpha a}^{\beta b}}$$

are unitary. Such a U is called a biunitary.

We represent the entries of U and $\widetilde{U}$ by positive and negative labeled crossings:

$$(2.1) \quad U_{\alpha a}^{\beta b} = \qquad \widetilde{U}_{\beta a}^{\alpha b} =$$

By definition of $\widetilde{U}$, the crossings above represent conjugate complex numbers.

Note that there are two types of strings, $[k]$-strings and $[p]$-strings, depending on the index set used. In a Jones diagram (such as the right-hand side of Figure 1, on the next page), the only allowed crossings are $[p]$-strings *over* $[k]$-strings. Strings are either bounded or unbounded; bits of string between crossings are considered bounded strings. A *state* on a diagram is a labeling of all strings from the appropriate index set, either $[k]$ or $[p]$. We evaluate a state by taking the product of all contributions from boxes and crossings. A diagram without any index labels is considered a function from its *boundary conditions* to $\mathbb{C}$. A boundary condition is a choice of indices for the unbounded strings, and it maps to the sum of the values of the states consistent with the boundary condition. For example, if $\gamma \in [p]$ and $b_i, a_i \in [k]$, $1 \le i \le 2n$, then

$$= \sum_{\substack{\gamma_0, \gamma_1, \ldots, \gamma_{2n} \in [p] \\ \gamma_0 = \gamma \\ \gamma_{2n} = \gamma'}} \prod_{0 \le i < n}$$

If $\alpha, \beta \in [p]$, the equation in Figure 1 translates as follows: for all $w \in [k]^{2n}$,

$$(2.2) \quad \delta_{\alpha,\alpha'} \cdot \delta_{\beta,\beta'} \cdot F_w = \sum_{x,y \in [k]^{2n}} F_y \cdot \quad \beta \xleftarrow{\quad} \beta' \quad \cdot \quad \alpha \xrightarrow{\quad} \alpha'$$

We say that $F \in \mathrm{End}(V^n)$ satisfies the relative commutant condition if for all $\alpha, \beta \in [p]$, Figure 1 holds. For $n = 1, 2, \ldots$, let

$$\widetilde{B}_n = \{F \in \mathrm{End}(V^n) : F \text{ satisfies the relative commutant condition}\}.$$

Set $\widetilde{B}_0 = \mathrm{End}(V^0) = \mathbb{C} \cdot 1$. For each $n \geq 2$ define $\widetilde{E}_{n-1} \in \widetilde{B}_n$ by

$$(\widetilde{E}_{n-1})_{a_1 a_2 \ldots a_{2n}} = \frac{1}{k} \delta_{a_{n-1}, a_n} \delta_{a_{n+1}, a_{n+2}}.$$

Theorem 2.1 (Ocneanu, Jones). *The sequence of $*$-algebras*

$$\mathbb{C} = \widetilde{B}_0 \subset \widetilde{B}_1 \subset \widetilde{B}_2 \subset \cdots \subset \widetilde{B}_n \subset \ldots$$

with the trace inherited from $(\mathrm{End}(V^n))_n$ and the Jones projections $\widetilde{E}_n$ is isomorphic to the tower of higher relative commutants of $R_U \subset R$ ([9]).
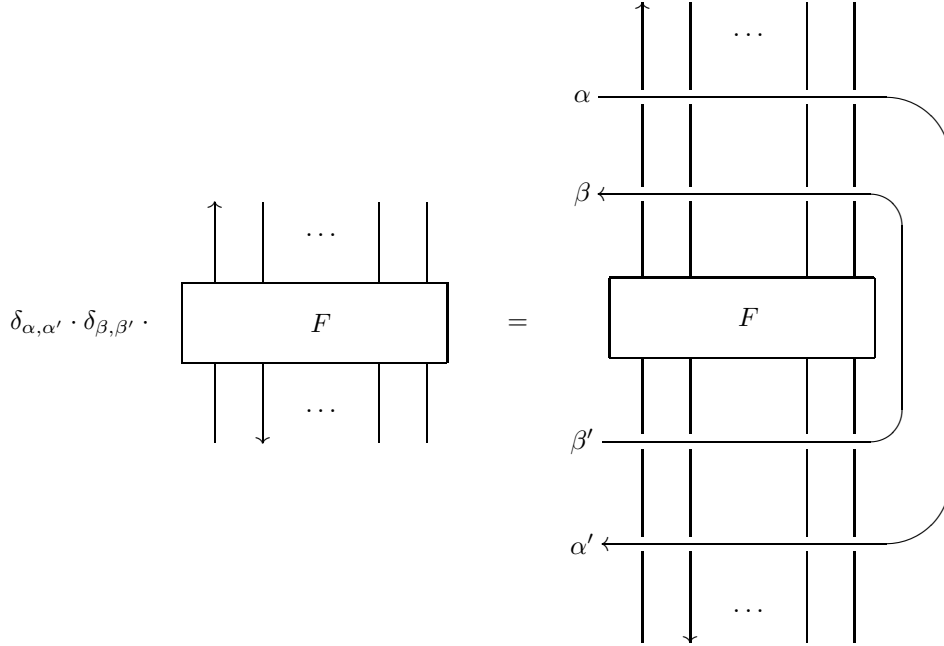

FIGURE 1. Relative commutant condition

3. The Group G

Fix a permutation biunitary $U \in M_p(\mathbb{C}) \otimes M_k(\mathbb{C})$, that is, a biunitary whose entries are 0's and 1's. We give formulations of the nth higher relative commutant of $R_U \subset R$ in Proposition 3.5 and Corollary 3.6. These formulations are a twist on the one given by Krishnan and Sunder in [10].

Let $\mathcal{F}$ be the set of words of finite length on the alphabet $[k]$. The following is Proposition 6 in [10].

Proposition 3.1 (Krishnan-Sunder). *For each $w \in \mathcal{F}$, there exists a unique permutation action on $[p]$*

$$w : [p] \to [p]$$
$$\alpha \mapsto w(\alpha)$$

and for each $\alpha \in [p]$ a unique permutation action on $\mathcal{F}$

$$\alpha : \mathcal{F} \to \mathcal{F}$$
$$w \mapsto \alpha(w)$$

such that

$$= \delta_{\alpha, w_1(\beta)} \cdot \delta_{w_2, \beta(w_1)}.$$

Remark 3.2. Noting that the crossings in (2.1) are either both equal to 1 or both equal to 0, we have the following recursive formulas for the actions defined in Proposition 3.1. Let $\beta \in [p]$ and let $a_1 a_2 \ldots a_r a_{r+1}$ be a word of length $r + 1 > 1$ in $\mathcal{F}$. Then

$$(3.1) \qquad a_1 a_2 \ldots a_{r+1}(\beta) = \begin{cases} a_{r+1}(a_1 a_2 \ldots a_r(\beta)), & \text{if } r \text{ is even,} \\ a_{r+1}^{-1}(a_1 a_2 \ldots a_r(\beta)), & \text{if } r \text{ is odd.} \end{cases}$$

Also, define $\gamma \in [p]$ by

$$(3.2) \qquad \gamma = \begin{cases} a_1 a_2 \ldots a_r(\beta), & \text{if } r \text{ is even,} \\ a_1 a_2 \ldots a_r a_{r+1}(\beta), & \text{if } r \text{ is odd.} \end{cases}$$

Then

$$(3.3) \qquad \beta(a_1 a_2 \ldots a_{r+1}) = \beta(a_1 a_2 \ldots a_r) \circ \gamma(a_{r+1})$$

where $\circ$ denotes concatenation of words.

Lemma 3.3. *If $y, x \in \mathcal{F}$ have the same length and $\alpha, \beta \in [p]$, then*

$$= \delta_{\beta, x(\alpha)} \cdot \delta_{y, \alpha(x)}.$$

Proof. Set

$$x' = \alpha^{-1}(y),$$
$$\beta' = x'(\alpha).$$

We have the following identity for Jones diagrams [9]:

$$(3.4) \quad \delta_{x,x'} \cdot \delta_{\beta,\beta'} \quad =$$

$$(3.5) \quad = \sum_{y',\alpha'}$$

By Proposition 3.1, the second factor of (3.5) equals

$$\delta_{\beta',x'(\alpha')}\delta_{y',\alpha'(x')} = \delta_{y,y'}\delta_{\alpha,\alpha'}.$$

Hence,

$$\delta_{x,x'} \cdot \delta_{\beta,\beta'} \quad = \quad$$

But

$$\delta_{x,x'} \cdot \delta_{\beta,\beta'} = \delta_{y,\alpha(x)} \cdot \delta_{\beta,x(\alpha)};$$

so we are done. $\square$

Action of $\mathcal{F}$ on $[p]^2$. For each $w \in \mathcal{F}$ define an action of w on $[p]^2 = [p] \times [p]$ by

$$w : [p]^2 \to [p]^2$$
$$(\alpha, \beta) \mapsto w(\alpha, \beta) = (w(\alpha), (\beta^{-1}\alpha(w))(\beta)).$$

Lemma 3.4. *Let $w \in \mathcal{F}$ and $\alpha, \beta \in [p]$. Then*

(3.6)

$$= \qquad \delta_{(\gamma,\epsilon),w(\alpha,\beta)} \cdot \delta_{x,\beta^{-1}\alpha(w)}.$$

Proof. Let n be the length of w. The left-hand side of (3.6) equals

$$= \sum_y \delta_{\gamma,w(\alpha)}\delta_{y,\alpha(w)}\delta_{y,\beta(x)}\delta_{\epsilon,x(\beta)} \qquad \text{(Lemma 3.3)}$$

$$= \delta_{(\gamma,\epsilon),w(\alpha,\beta)}\delta_{x,\beta^{-1}\alpha(w)}.$$

$\square$

The following proposition and corollary reformulate Lemma 8 and part of Proposition 10 in [10].

Proposition 3.5. *The following are necessary and sufficient conditions for $F \in \mathrm{End}(V^n)$ to satisfy the relative commutant condition. For all $\alpha, \beta \in [p]$ and $w \in \mathcal{F}$ of length $2n$,*

(1) *if $F_w \neq 0$, then w acts identically on $[p]^2$;*
(2) $F_w = F_{\beta^{-1}\alpha(w)}$.

Proof. Using the relative commutant condition in Figure 1 and unwrapping the unitary lines, we obtain the equivalent condition: for all $\alpha, \alpha', \beta, \beta' \in [p]$,

(3.7) $\qquad \delta_{\alpha,\alpha'} \cdot \delta_{\beta,\beta'} \cdot F_w = \displaystyle\sum_{w'}$

$\qquad \cdot \quad F_{w'}.$

By Lemma 3.4, (3.7) is equivalent to

(3.8) $\qquad\qquad\qquad \delta_{\alpha,\alpha'} \cdot \delta_{\beta,\beta'} \cdot F_w = \delta_{(\alpha',\beta'),w(\alpha,\beta)}F_{\beta^{-1}\alpha(w)}.$

The conditions of the proposition clearly imply (3.8) for all $\alpha, \alpha', \beta, \beta', w$. We must prove necessity of the conditions, assuming (3.8) holds for all α, β, w. Set $\alpha' = \alpha$ and $\beta' = \beta$. Then the first condition is obvious. The second condition is obvious if $w(\alpha, \beta) = (\alpha, \beta)$. If $w(\alpha, \beta) \neq (\alpha, \beta)$, then $F_w = 0$, and by symmetry

$$F_{\beta^{-1}\alpha(w)} = \delta_{(\beta,\alpha),\beta^{-1}\alpha(w)(\beta,\alpha)} F_w.$$

Therefore, $F_{\beta^{-1}\alpha(w)} = 0$ as well. $\square$

The Group of $R_U \subset R$. As in [10], define G to be the subgroup of permutations of $\mathcal{F}$ generated by the elements $\beta^{-1}\alpha$, for $\alpha, \beta \in [p]$.

Corollary 3.6. *The following are necessary and sufficient conditions for $F \in \text{End}(V^n)$ to satisfy the relative commutant condition. For all $w \in \mathcal{F}$ of length $2n$,*

   (1) *if $F_w \neq 0$, then for all $g \in G$, $g(w)$ acts identically on $[p]^2$;*
   (2) *for all $g \in G$, $F_w = F_{g(w)}$.*

Proof. It is easy to see that these conditions are equivalent to the conditions of the previous proposition. $\square$

Remark 3.7. By the definition of the action of $\mathcal{F}$ on $[p]^2$, $g(w)$ acts identically on $[p]^2$ for all $g \in G$ iff $g(w)$ acts identically on $[p]$ for all $g \in G$.

4. Notation for Graphs

We define a graph $\mathcal{H} = (V, E)$ to be a pair of sets V and E, the vertex set and the edge set respectively, such that to every edge $e \in E$ there are associated two vertices in V, called its endpoints. Thus our graphs have unoriented edges and allow multiple edges. In fact, all the graphs we consider are bipartite, so in particular we will not consider graphs with loops. A homomorphism of graphs

$$\phi : \mathcal{H}_1 = (V_1, E_1) \rightarrow \mathcal{H}_2 = (V_2, E_2)$$

consists of maps of sets $\phi_V : V_1 \rightarrow V_2$ and $\phi_E : E_1 \rightarrow E_2$ such that if $\phi_E : e \mapsto f$, then ϕ_V maps the endpoints of e onto the endpoints of f.

A *path* ρ on $\mathcal{H}$ of length n is a finite alternating sequence of vertices and edges, $v_0 e_1 v_1 e_2 \ldots e_n v_n$, where $v_i \in V$, $0 \leq i \leq n$, and $e_i \in E$, $1 \leq i \leq n$, such that for each i the set of endpoints of e_i is $\{v_{i-1}, v_i\}$. We say that e_i is *between* v_{i-1} and v_i. We say that ρ starts at v_0 and ends at v_n. Denote by ρ^t the path

$$v_n e_n v_{n-1} e_{n-1} \ldots e_1 v_0.$$

If $\mathcal{H}$ has a distinguished root vertex, then the path ρ is a *rooted path* if it starts at the root. We will denote the root vertex of a graph by $*$. If ρ_1 and ρ_2 are two paths

and ρ_2 starts at the vertex at which ρ_1 ends, we denote the concatenation of ρ_1 and ρ_2 by $\rho_1 \circ \rho_2$. The distance between two vertices v and w is the minimum path length over all paths starting at v and ending at w. Vertices v and w are *neighbors* if the distance between them is 1.

In Section 9, we will want to treat graphs as topological spaces. A graph $\mathcal{H} = (V, E)$ can be represented by a topological space $\mathcal{H}^r$ as defined in [6], p. 17. Each vertex is represented by a distinct point and each edge is represented by a distinct arc homeomorphic to the closed interval $[0, 1]$. The boundary points of an arc represent the endpoints of the corresponding edge. The interiors of the arcs are mutually disjoint and do not meet the points representing vertices. If $\phi : \mathcal{H}_1 \to \mathcal{H}_2$ is a graph morphism, then the induced topological map $\phi^r : \mathcal{H}_1^r \to \mathcal{H}_2^r$ is continuous.

Given a vertex v of $\mathcal{H}$, let $E_{\mathcal{H}}(v)$ denote the set of edges which have v as an endpoint. Following [3], we call a graph morphism $\phi : \mathcal{H}_1 \to \mathcal{H}_2$ a *local isomorphism* if for all vertices v in $\mathcal{H}_1$, the restriction of ϕ to $E_{\mathcal{H}_1}(v)$ is a bijection of $E_{\mathcal{H}_1}(v)$ with $E_{\mathcal{H}_2}(\phi(v))$.

Lemma 4.1. $\phi^r : \mathcal{H}_1^r \to \mathcal{H}_2^r$ *is a covering map if* $\phi : \mathcal{H}_1 \to \mathcal{H}_2$ *is a surjective local isomorphism.*

Proof. See [6], p. 75. A similar result is proved in [3]. $\qquad\square$

5. The Graph $\mathcal{H}$

An important ingredient of Krishnan and Sunder's analysis is an equivalence relation on words in $\mathcal{F}$. The main point of this section is to encode that relation into a graph $\mathcal{H}$. For this purpose it is more convenient to define a superficially different relation in Definition 5.3, but we prove in Proposition 5.6 that the two relations are essentially the same.

Lemma 5.1. *Let* $w_1, w_2 \in \mathcal{F}$ *be words of the same length and let* $\alpha, \beta, \gamma, \delta \in [p]$. *Then*

(5.1)

Proof. Rotating the right-hand side of (5.1) by 180 degrees, we obtain
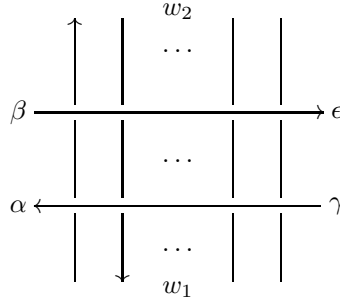
which equals (using Lemma 3.3 and Proposition 3.1 to deduce (5.3))

(5.2)
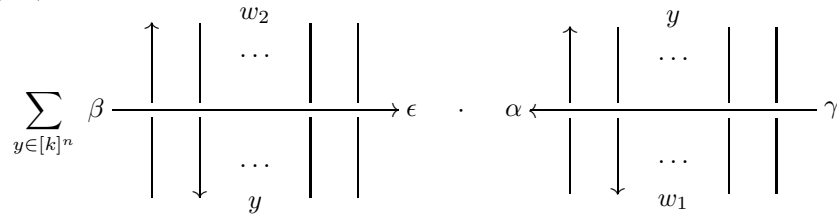
(5.3)
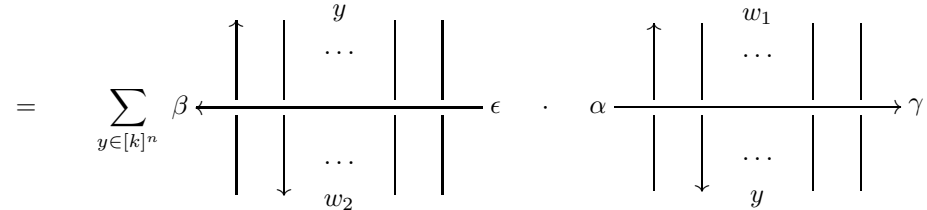
which equals the left-hand side of (5.1).   □

Lemma 5.2. *Assume $w_1, w_2 \in \mathcal{F}$ and $w_1 w_2^t$ has even length.*

(1) *Let $\alpha, \beta \in [p]$ and set $g = \beta^{-1}\alpha \in G$. If $w_1 w_2^t$ acts identically on $[p]^2$, then $g(w_1 w_2^t) = g(w_1)g(w_2)^t$.*

(2) *w_1 and w_2 have the same action on $[p]^2$ if and only if $w_1 w_2^t$ acts identically on $[p]^2$.*

Proof. Fix any $\alpha, \beta \in [p]$. Let $(\gamma, \epsilon) = w_1(\alpha, \beta)$, and let x_1 and x_2 be words in $\mathcal{F}$ of the same length as w_1 and w_2, respectively. We have (using Lemma 3.4 to deduce

(5.5) and Lemma 5.1 to deduce (5.6))

$$\text{(5.4)}$$

$$\text{(5.5)} \quad =$$

$$\text{(5.6)} \quad =$$

Assume that w_1 and w_2 have the same action on $[p]^2$, and set $x_1 = \beta^{-1}\alpha(w_1)$ and $x_2 = \beta^{-1}\alpha(w_2)$. By Lemma 3.4, (5.6) equals 1. Applying Lemma 3.4 to (5.4), we obtain that $w_1 w_2^t$ acts identically on (α, β). Since (α, β) was arbitrary, this proves direction ($\implies$) of part 2.

Now assume that $w_1 w_2^t$ acts identically on $[p]^2$. Let x_1 and x_2 be such that $x_1 x_2^t = \beta^{-1}\alpha(w_1 w_2^t)$. By Lemma 3.4, (5.4) equals 1. Applying Lemma 3.4 to (5.6), we obtain $w_2(\alpha, \beta) = (\gamma, \epsilon)$, $x_1 = \beta^{-1}\alpha(w_1)$, and $x_2 = \beta^{-1}\alpha(w_2)$. Then $w_1(\alpha, \beta) = w_2(\alpha, \beta)$ for arbitrary (α, β), which finishes the proof of part 2. And $g(w_1 w_2^t) = g(w_1)g(w_2)^t$, which proves part 1. $\qquad\square$

The following relation is essentially the same as the one defined by Krishnan and Sunder in [10], as we shall see in Proposition 5.6.

Definition 5.3. Define the relation $\sim$ on $\mathcal{F}$ by $w_1 \sim w_2$ if $w_1 w_2^t$ is of even length and if for all $g \in G$, $g(w_1 w_2^t)$ acts identically on $[p]^2$. We may also write $w_1 w_2^t \sim e$, where e denotes the empty word.

Lemma 5.4. *The set* $\{w \in \mathcal{F} : w \sim e\}$ *is stabilized by* G.

Proof. Obvious. $\qquad\square$

Lemma 5.5. *Let $w_1, w_2 \in \mathcal{F}$. If $w_1 \sim w_2$, then for all $g \in G$, $g(w_1 w_2^t) = g(w_1)g(w_2)^t$.*

Proof. Using Lemma 5.4 and Lemma 5.2, this is an exercise in induction on the number of generators needed to generate g. $\qquad\square$

Proposition 5.6. *For all $w_1, w_2 \in \mathcal{F}$, $w_1 \sim w_2$ if and only if $w_1 w_2^t$ has even length and for all $g \in G$, $g(w_1)$ and $g(w_2)$ have the same action on $[p]^2$.*

Proof. Direction ($\Rightarrow$) follows immediately from Lemma 5.5, Lemma 5.4, and part 2 of Lemma 5.2. To prove the other direction, assume that for all $g \in G$, $g(w_1)$ and $g(w_2)$ have the same action on $[p]^2$. Then for all g, $g(w_1)g(w_2)^t$ acts identically on $[p]^2$ by Lemma 5.2. And $g^{-1}(g(w_1)g(w_2)^t) = w_1 w_2^t$ by the first part of Lemma 5.2; hence $g(w_1 w_2^t) = g(w_1)g(w_2)^t$. Therefore $g(w_1 w_2^t)$ acts identically on $[p]^2$ for all $g \in G$, so $w_1 \sim w_2$. $\qquad\square$

Corollary 5.7. *The relation $\sim$ is an equivalence relation.*

Proof. Immediate from Proposition 5.6. $\qquad\square$

The Graph of $R_U \subset R$.   We can now construct the rooted graph associated to the Krishnan-Sunder subfactor $R_U \subset R$. The graph $\mathcal{H} = (V_U, E_U)$ associated to $R_U \subset R$ has vertex set $V_U = \mathcal{F}/\sim$, the set of equivalence classes of $\mathcal{F}$. We denote by $[w]$ the equivalence class of $w \in \mathcal{F}$. The edge set of $\mathcal{H}$ is

$$E_U = \{([w_u], [w_v]\}, a) : w_u, w_v \in \mathcal{F},\, a \in [k],\, \text{and } w_u a w_v^t \sim e\}$$

which we show to be well-defined in the following lemma. The endpoints of the edge $(\{u, v\}, a)$ in E_U are u and v. The root of $\mathcal{H}$ is the equivalence class of the empty word e. Note that $\mathcal{H}$ is bipartite: the even vertices are those classes whose representatives have even length. Define the weight vector τ on $\mathcal{H}$ by $\tau_u = 1$ for every vertex u of $\mathcal{H}$. We shall see in Remark 5.9 that τ_U is an eigenvector of $\mathcal{H}$ with eigenvalue k.

Lemma 5.8. *The condition $w_u a w_v^t \sim e$ is independent of the choice of representatives w_u and w_v and is symmetric in u and v.*

Proof. The relation $w_u a w_v^t \sim e$ is equivalent to $w_u \sim w_v a$ and to $w_u a \sim w_v$. Hence the relation is independent of the choice of representatives. By symmetry of $\sim$ (Corollary 5.7), $w_u a w_v^t \sim e \implies (w_u a w_v^t)^t \sim e$; hence $w_v a w_u^t \sim e$. Therefore the relation $w_u a w_v^t \sim e$ is symmetric in u and v. $\qquad\square$

Remark 5.9. By the proof of Lemma 5.8 it is evident that every vertex u of $\mathcal{H}$ is the endpoint of exactly k edges, namely $\{(\{u, [w_u a]\}, a)\}_{a \in [k]}$, where w_u is any representative of u. It follows that τ is an eigenvector of $\mathcal{H}$ with eigenvalue k.

Remark 5.10. By Proposition 5.6, the class of $g(w)$, $w \in \mathcal{F}$, is independent of the choice of $w \in [w]$.

Two Extreme Cases.   In [10], Krishnan and Sunder describe two extreme cases: when the action of $[p]$ on $[k]$ is trivial and when the action of $[k]$ on $[p]$ is trivial. In the first case, our definition of $\mathcal{H}$ coincides with Krishnan and Sunder's description of the principal graph, a Cayley-type graph of the subgroup of S_p generated by the $[k]$-action. In the second case, it is evident that $\mathcal{H}$ is as trivial as possible; it is the bipartite graph with two vertices and k edges.

6. The Action of G on $\mathcal{H}$

We construct an action of G on $\mathcal{H}$, by which we mean a group homomorphism from G to the root preserving automorphisms of $\mathcal{H}$. The homomorphism we construct is actually injective, but we delay proving this until Section 7.

Proposition 6.1. *For any g in G and edge $(\{u, v\}, a)$ in E_U, there exists a unique b in $[k]$ such that for all $w_u \in u$ and $w_v \in v$, we have*

$$g(w_u a w_v^t) = g(w_u) b g(w_v)^t.$$

Proof. It is easy to see by induction on length that it suffices to prove the result for a set of generators of G. We prove the result for the generating set $\{\beta^{-1}\alpha\}_{\alpha, \beta \in [p]}$ of G.

Fix $g = \beta^{-1}\alpha$, $\alpha, \beta \in [p]$. We claim the conclusion of the proposition is symmetric in u and v. Consider

$$(6.1) \qquad\qquad g(w_u a w_v^t) = g(w_u) b g(w_v)^t$$

$$(6.2) \qquad\qquad g(w_u a) g(w_v)^t = g(w_u) b g(w_v)^t$$

$$(6.3) \qquad\qquad g(w_u a) = g(w_u) b$$

$$(6.4) \qquad\qquad g(w_v) g(w_u a)^t = g(w_v) b g(w_u)^t$$

$$(6.5) \qquad\qquad g(w_v a w_u^t) = g(w_v) b g(w_u)^t$$

Then (6.1) $\iff$ (6.2) (by Lemma 5.5) $\iff$ (6.3) $\iff$ (6.4) $\iff$ (6.5) (by Lemma 5.5), which proves the claim. So we may assume without loss of generality that u is the even vertex.

Let

$$(\gamma_u, \epsilon_u) = w_u(\alpha, \beta) \quad \text{and} \quad (\gamma_v, \epsilon_v) = w_v(\alpha, \beta).$$

By Proposition 5.6, γ_u, ϵ_u, γ_v, and ϵ_v are independent of the choice of w_u and w_v. Let

$$x_u = g(w_u) \quad \text{and} \quad x_v = g(w_v).$$

Then

$$g(w_u a w_v^t) = x_u g(w_v a)^t = g(w_u a) x_v$$

by Lemma 5.2. Hence, there is a $b \in [k]$ such that

$$g(w_u a w_v^t) = x_u b x_v.$$

Then (using Lemma 5.1 and Lemma 3.4 for the third equality and Lemma 3.4 for the fourth equality)

$$
1 \quad = \quad
\begin{array}{c}
w_u a w_v^t \\
\cdots \\
\alpha \longrightarrow \alpha \\
\cdots \\
\beta \longleftarrow \beta \\
\cdots \\
x_u b x_v^t
\end{array}
$$

$$
= \sum_{\substack{\gamma_u', \epsilon_u', \gamma_v', \\ \epsilon_v' \in [p]}}
\begin{array}{c}
w_u \\
\cdots \\
\alpha \longrightarrow \gamma_u' \\
\cdots \\
\beta \longleftarrow \epsilon_u' \\
\cdots \\
x_u
\end{array}
\cdot
\begin{array}{c}
a \\
\gamma_u' \longrightarrow \gamma_v' \\
\epsilon_u' \longleftarrow \epsilon_v' \\
b
\end{array}
\cdot
\begin{array}{c}
w_v^t \\
\cdots \\
\gamma_v' \longrightarrow \alpha \\
\cdots \\
\epsilon_v' \longleftarrow \beta \\
\cdots \\
x_v^t
\end{array}
$$

$$
=
\begin{array}{c}
w_u \\
\cdots \\
\alpha \longrightarrow \gamma_u \\
\cdots \\
\beta \longleftarrow \epsilon_u \\
\cdots \\
x_u
\end{array}
\cdot
\begin{array}{c}
a \\
\gamma_u \longrightarrow \gamma_v \\
\epsilon_u \longleftarrow \epsilon_v \\
b
\end{array}
\cdot
\begin{array}{c}
w_v \\
\cdots \\
\alpha \longrightarrow \gamma_v \\
\cdots \\
\beta \longleftarrow \epsilon_v \\
\cdots \\
x_v
\end{array}
$$

$$
= \delta_{x_u, g(w_u)} \delta_{b, \epsilon_u^{-1} \gamma_u(a)} \delta_{(\gamma_v, \epsilon_v), a(\gamma_u, \epsilon_u)} \delta_{x_v, g(w_v)}.
$$

Therefore, $b = \epsilon_u^{-1} \gamma_u(a)$. $\qquad \square$

The Group Action. Given any $g \in G$, $[w] \in V_U$, and $(\{u, v\}, a) \in E_U$, define $g_{\mathcal{H}}([w]) = [g(w)]$ and $g_{\mathcal{H}}((\{u, v\}, a)) = (\{g_{\mathcal{H}}(u), g_{\mathcal{H}}(v)\}, b)$, where b is as in Proposition 6.1. By Proposition 5.6 it's clear that $[g(w)]$ is independent of the choice of $w \in [w]$. It follows from Proposition 6.1 that $(\{g_{\mathcal{H}}(u), g_{\mathcal{H}}(v)\}, b)$ is an edge in E_U. Therefore $g_{\mathcal{H}}$ is a well-defined graph endomorphism of $\mathcal{H}$. It is evident that $g_{\mathcal{H}}$ preserves the root vertex, and it is an easy exercise to show that $g_{\mathcal{H}} h_{\mathcal{H}} = (gh)_{\mathcal{H}}$ for any $g, h \in G$. Therefore, we have a group homomorphism

$$
G \to \operatorname{Aut}(\mathcal{H})
$$
$$
g \mapsto g_{\mathcal{H}}
$$

from G to the group of root-preserving automorphisms of $\mathcal{H}$. When there is no ambiguity, we shall simply write g instead of $g_{\mathcal{H}}$.

7. Higher Relative Commutants of $R_U \subset R$

The description of the higher relative commutants given in Corollary 3.6 can be rephrased in terms of the *path algebras* ([13], [18], [4]) on $\mathcal{H}$ and the action of G on $\mathcal{H}$. We briefly recall some ideas concerning path algebras. Let $V_\mathcal{H}^n$ denote the complex vector space with orthonormal basis the set of rooted paths of length n on $\mathcal{H}$. If $F \in \mathrm{End}(V_\mathcal{H}^n)$, write F in matrix form $(F_{\rho_1,\rho_2})_{\rho_1,\rho_2}$, where ρ_1 and ρ_2 range over the rooted paths of length n on $\mathcal{H}$. The nth path algebra P_n on $\mathcal{H}$ is the subalgebra of $\mathrm{End}(V_\mathcal{H}^n)$,

$$P_n = \{F \in \mathrm{End}(V_\mathcal{H}^n) : \ F_{\rho_1,\rho_2} = 0, \text{ if } \rho_1 \text{ and } \rho_2 \text{ end at different vertices}\}$$

If for each vertex v of $\mathcal{H}$ we let $X_v^n \subset V_\mathcal{H}^n$ denote the space spanned by paths ending at v, then

$$P_n = \bigoplus_{v \in \mathcal{H}} \mathrm{End}(X_v^n).$$

Note that X_v may be 0.

Since F stabilizes the end vertices of paths, there is a natural inclusion map $i : P_n \to P_{n+1}$, where $i(F)$ preserves the part of a path after its nth vertex. More precisely, if

$$\mu_1 = \rho_1 \circ \nu_1,$$
$$\mu_2 = \rho_2 \circ \nu_2$$

where ρ_i has length n and ν_i has length 1, $i = 1, 2$, then

$$i(F)_{\mu_1,\mu_2} = \delta_{\nu_1,\nu_2} F_{\rho_1,\rho_2}.$$

Thus we have a tower of finite dimensional C^*-algebras

$$\mathbb{C} = P_0 \subset P_1 \subset \dots P_n \subset \dots.$$

Let tr_n be the trace on P_n such that the trace of a minimal projection in $\mathrm{End}(X_v^n)$ is $\frac{1}{k^n}$ times the weight of τ on v (which is 1 for all v). Since τ is a k-eigenvector, the traces tr_n are compatible with the inclusion maps, and we obtain a trace tr on $\bigcup_n P_n$ with $\mathrm{tr}(1) = 1$.

If ρ is a rooted path of length n, let $\rho(i)$ denote the ith edge of ρ. Define $E_{n-1} \in P_n$, $n > 1$, by

$$(E_{n-1})_{\rho_1,\rho_2} = \frac{1}{k} \delta_{\rho_1(n-1),\rho_1(n)} \delta_{\rho_2(n-1),\rho_2(n)}$$

if ρ_1 and ρ_2 end at the same vertex. Then $(E_n)_n$ is a sequence of Jones projections with modulus k, and tr satisfies the Markov property with respect to $(E_n)_n$. We can sum this up by saying that $\mathcal{H}$ with weight vector τ is the principal graph of the tower $(P_n)_n$.

There is a natural group action of G on $V_\mathcal{H}^n$ for each n, arising from the natural permutation action of $\mathrm{Aut}(\mathcal{H})$ on the set of rooted paths of length n. Namely, if $g \in \mathrm{Aut}(\mathcal{H})$ and $v_0 e_1 v_1 \dots e_n v_n$ is a rooted path, then

$$g(v_0 e_1 v_1 \dots e_n v_n) = g(v_0) g(e_1) g(v_1) \dots g(e_n) g(v_n).$$

For each n let B_n be the subalgebra of P_n that commutes with the action of G. It is easy to check that $B_n \subset B_{n+1}$ for all n.

Theorem 7.1. *There is a trace-preserving $*$-isomorphism from the tower of higher relative commutants of $R_U \subset R$ to the tower of algebras $(B_n)_n$. The isomorphism preserves Jones projections.*

The rest of this section is devoted to proving the theorem. By Remark 5.9, if $a_1 a_2 \ldots a_n \in \mathcal{F}$, there is a unique rooted path $v_0 e_1 v_1 \ldots e_n v_n$ on $\mathcal{H}$ such that $e_i = (\{v_{i-1}, v_i\}, a_i)$, $1 \le i \le n$. Denote this path $\phi(a_1 \ldots a_n)$. It is easy to see that

$$\phi : \mathcal{F} \to \{\text{Rooted paths on } \mathcal{H}\}$$
$$w \mapsto \phi(w)$$

is a bijection of sets. Thus ϕ extends to a unitary vector space isomorphism of V^n with $V_{\mathcal{H}}^n$, which we also denote by ϕ.

Lemma 7.2. *If $\phi(a_1 a_2 \ldots a_n) = v_0 e_1 v_1 \ldots e_n v_n$, then $v_i = [a_1 \ldots a_i]$, $1 \le i \le n$.*

Proof. We prove the lemma by induction. Since $v_0 = [e]$, it suffices to show that if $v_{i-1} = [w]$ then $v_i = [wa_i]$, $1 \le i \le n$. This follows from the fact that $e_i = (\{v_{i-1}, v_i\}, a_i)$ is an edge. $\square$

The lemma immediately implies the following corollary.

Corollary 7.3. *If $w \in \mathcal{F}$, then $w \sim e$ if and only if $\phi(w)$ is a closed path.*

Proposition 7.4. *If $g \in G$ and $w \in \mathcal{F}$, then*

$$\phi(g(w)) = g(\phi(w)).$$

Proof. Let $w = a_1 \ldots a_n$, where $a_i \in [k]$ for $1 \le i \le n$. Let $\phi(w) = *e_1 v_1 \ldots e_n v_n$, where $*$ denotes the root. We prove the result by induction on n. The result is trivial when $n = 0$. Let $w' = a_1 \ldots a_{n-1}$. Clearly $\phi(w') = *e_1 v_1 \ldots e_{n-1} v_{n-1}$. By the induction hypothesis, we may assume that $g(\phi(w')) = \phi(g(w'))$. Thus, there exist b_i, u_i, f_i, $1 \le i \le n-1$, such that $g(w') = b_1 \ldots b_{n-1}$ and $g(*e_1 v_1 \ldots e_{n-1} v_{n-1}) = *f_1 u_1 \ldots f_{n-1} u_{n-1}$ where $f_i = (\{u_{i-1}, u_i\}, b_i)$ for each i. By Lemma 7.2, $u_{n-1} = [w']$. By definition, $g(e_n)$ is the edge $(\{[g(w')], [g(w)]\}, b)$ such that $g(w' a_n w^t) = g(w') b g(w)^t$. So we have

$$g(w' a_n w^t) = g(w w^t)$$
$$= g(w) g(w)^t \qquad \text{(Lemma 5.5).}$$

Therefore, $g(w) = g(w')b$. So we have

$$g(\phi(w)) = g(\phi(w')) \circ g(v_{n-1} e_n v_n)$$
$$= \phi(g(w')) \circ g(v_{n-1} e_n v_n)$$
$$= \phi(g(w')b)$$
$$= \phi(g(w)).$$

$\square$

Since ϕ is an isomorphism, the previous proposition implies the following corollary.

Corollary 7.5. *The homomorphism $G \to \text{Aut}(\mathcal{H})$ is injective.*

Proof of Theorem 7.1. Suppose $F \in \mathrm{End}(V^n)$. Then $\phi F \phi^{-1} \in \mathrm{End}(V^n_{\mathcal{H}})$, and if $w_1, w_2 \in \mathcal{F}$ are of length n, then

$$(7.1) \qquad F_{w_1 w_2^t} = (\phi F \phi^{-1})_{\phi(w_1), \phi(w_2)}.$$

We first prove that the injective $*$-homomorphism

$$\mathrm{End}(V^n) \to \mathrm{End}(V^n_{\mathcal{H}})$$
$$F \mapsto \phi F \phi^{-1}$$

maps $\widetilde{B}_n$ onto B_n. We claim that F satisfies the first condition of Corollary 3.6 iff $\phi F \phi^{-1} \in P_n$. If w_1, w_2 are as above, then $w_1 w_2^t \sim e \iff w_1 \sim w_2 \iff \phi(w_1)$ and $\phi_{(}w_2)$ end at the same vertex (by Lemma 7.2). Since $w \sim e$ is equivalent to w satisfying the first condition of Corollary 3.6, the claim follows.

Now assume that F satisfies the first condition of Corollary 3.6. Then we claim that F satisfies the second condition of the corollary iff $\phi F \phi^{-1}$ commutes with the G-action. Note that $F' \in P_n$ commutes with the G-action iff $F'_{\rho_1, \rho_2} = F'_{g(\rho_1), g(\rho_2)}$ for all rooted paths ρ_1, ρ_2 of length n and $g \in G$. Hence it suffices to prove that if $w_1, w_2 \in \mathcal{F}$ are of length n and $g \in G$, then

$$(7.2) \qquad F_{g(w_1 w_2^t)} = (\phi F \phi^{-1})_{g\phi(w_1), g\phi(w_2)^t}.$$

Note that $w_1 w_2^t \sim e \iff g(w_1 w_2^t) \sim e$, and that $\phi(w_1)$ and $\phi(w_2)$ end at the same vertex iff $g(\phi(w_1))$ and $g(\phi(w_2))$ end at the same vertex. Thus, if $w_1 \not\sim w_2$, then both sides of (7.2) are 0, since F satisfies the first condition of Corollary 3.6. So assume $w_1 \sim w_2$. Then (7.2) follows from Proposition 7.4, Lemma 5.5, and (7.1). It follows from the two claims that $\phi \widetilde{B}_n \phi^{-1} = B_n$.

It is easy to see that the isomorphism of $\widetilde{B}_n$ with B_n implemented by ϕ respects the inclusion maps, preserves the trace, and maps $\widetilde{E}_n$ to E_n, for each n. The proof then follows from Theorem 2.1. $\qquad \square$

8. Principal Graph of $R_U \subset R$

For each vertex $v \in \mathcal{H}$, let G_v be the subgroup of G that fixes the vertex v. Consider the action of G on the vertex set of $\mathcal{H}$, and for each G-orbit fix a representative vertex w. Let O denote the set of representative vertices. Denote the orbit of w by $o(w)$. For every $w \in O$ and $u \in o(w)$, fix $g_u \in G$ satisfying $g_u(u) = w$. For any group H, let $\hat{H}$ denote the finite dimensional irreducible representations of H. Each g_u induces the isomorphism $\mathrm{Ad}^*_{g_u} : \hat{G}_w \to \hat{G}_u$, which we denote by $\hat{g_u}$,

$$\hat{g_u} : \hat{G}_w \to \hat{G}_u.$$

If u and w are vertices of $\mathcal{H}$, let $E_{u,w}$ be the vector space with basis the set of edges between u and w. $E_{u,w}$ is naturally a $(G_u \cap G_w)$-module.

Definition 8.1. Define the graphs Γ^0 and Γ as follows. The vertex set of Γ^0 is $\amalg_{w \in O} \hat{G}_w$, the disjoint union of the sets $\hat{G}_w$. Let $\lambda_v \in \hat{G}_v$ and $\lambda_w \in \hat{G}_w$, for $v, w \in O$, be vertices of Γ^0. Consider the action of G_w on $o(v)$, and let S be a set of representatives of the G_w orbits. The number of edges between λ_w and λ_v is

$$(8.1) \qquad \sum_{u \in S} \dim(\mathrm{Hom}(\lambda_w, \mathrm{Ind}^{G_w}_{G_u \cap G_w}((\mathrm{Res}^{G_u}_{G_u \cap G_w} \hat{g_u}(\lambda_v)) \otimes E_{u,w}))).$$

Let τ_Γ be the vector of weights on Γ^0 given by $\lambda_w \mapsto \dim \lambda_w \cdot c$, where if $\lambda_w \in \hat{G}_w$ then c is the order of $o(w)$. Let 1_* denote the trivial representation of G_*. Define Γ to be the connected component of Γ^0 containing the vertex 1_*.

A simple calculation involving Frobenious reciprocity shows that (8.1) is symmetric in λ_v and λ_w, so Γ^0 and Γ are well-defined.

Theorem 8.2. *The graph Γ with weight vector τ_Γ and distinguished vertex 1_* is the principal graph of $R_U \subset R$.*

The proof follows Lemma 8.7. The following remark applies to examples discussed later.

Remark 8.3. Suppose the following assumptions hold:
  (1) For every vertex $w \in \mathcal{H}$ and every G-orbit $o \subset \{$vertices of $\mathcal{H}\}$, G_w acts transitively on $\{u \in o : u$ is a neighbor of $w\}$.
  (2) $\mathcal{H}$ does not have multiple edges (distinct edges of $\mathcal{H}$ have distinct endpoint sets).

Then the edge formula (8.1) simplifies to

$$(8.2) \qquad \dim \left(\mathrm{Hom} \left(\lambda_w, \mathrm{Ind}_{G_u \cap G_w}^{G_w} \left(\mathrm{Res}_{G_u \cap G_w}^{G_u} \hat{g}_u(\lambda_v) \right) \right) \right)$$

where u is any vertex in $o(v)$ that neighbors w. If such an edge does not exist, then there are no edges between λ_v and λ_w.

Remark 8.4. It is easy to see that our description of the principal graph coincides with Krishnan and Sunder's in the two extreme cases discussed at the end of Section 5. If G is trivial, then Γ is identical to $\mathcal{H}$. On the other hand, suppose $\mathcal{H}$ is the bipartite graph with two vertices $\{0, 1\}$ and k edges. Note that G fixes both vertices. Hence, Γ^0 is the connected graph whose vertices are the finite dimensional irreducible G-representations and where the number of edges between two vertices, λ_1 and λ_2, is the multiplicity of λ_1 in $\lambda_2 \otimes E_{0,1}$. Then Γ is the connected component containing the trivial representation as a vertex.

By Theorem 7.1, the principal graph of $R_U \subset R$ is calculated from the tower $(B_n)_n$. For each nonnegative integer n and vertex $v \in \mathcal{H}$, let $p_v^n \in \mathrm{End}(V_{\mathcal{H}}^n)$ be the orthogonal projection to the span of paths ending at v. Recall that G acts on $V_{\mathcal{H}}^n$ by its action on paths of length n. Let C_n be the subalgebra of $\mathrm{End}(V_{\mathcal{H}}^n)$ generated by $\{p_v^n\}_{v \in \mathcal{H}}$ and the G-action on $V_{\mathcal{H}}^n$. Then, by definition,

$$(8.3) \qquad\qquad B_n = C_n' \cap \mathrm{End}(V_{\mathcal{H}}^n).$$

Thus, the simple summands of B_n are in bijection with the irreducible C_n-modules in $V_{\mathcal{H}}^n$. Let $\mathcal{M}_n$ be the set of isomorphism classes of irreducible C_n-modules contained in $V_{\mathcal{H}}^n$. Note that any C_n-module is naturally a G-module.

For each $w \in O$,

$$q_w = \sum_{u \in o(w)} p_u^n$$

is clearly a central (possibly 0) projection in C_n, and $\{q_w\}_{w \in O}$ is a partition of unity. Hence, for any $M \in \mathcal{M}_n$, there is a unique $w \in O$ such that $q_w M \neq 0$, or equivalently, $q_w M = M$. Let $\mathcal{M}_{w,n}$ denote the subset

$$\mathcal{M}_{w,n} = \{M \in \mathcal{M}_n : q_w M = M\}.$$

So $\mathcal{M}_n = \amalg_{w \in O} \mathcal{M}_{w,n}$. Also, let $\Lambda_{w,n} \subset \hat{G_w}$ be the set of irreducible G_w representations contained in $p_w^n V_{\mathcal{H}}^n$.

Lemma 8.5. *For each nonnegative integer n and $w \in O$, the map*

$$\mathcal{M}_{w,n} \longleftrightarrow \Lambda_{w,n}$$
$$M \mapsto p_w^n M$$

is a bijection and $\dim(M) = \dim(p_w^n M) \cdot c$, *where c is the order of $o(w)$.*

Proof. Since n is fixed, let $p_v = p_v^n$, for each $v \in \mathcal{H}$. Recall the elements $g_u \in G$, $u \in o(w)$, defined in the discussion before Definition 8.1. Clearly $\{p_v g_v^{-1} g_u p_u\}_{v,u \in o(w)}$ is a set of matrix units for the partition $\{p_u\}_{u \in o(w)}$ of q_w. Let V_w be the complex vector space with orthonormal basis indexed by the G-orbit of w. The matrix units give rise to isomorphisms

$$q_w C_n \longleftrightarrow \mathrm{End}(V_w) \otimes p_w C_n p_w$$
$$q_w V_{\mathcal{H}}^n \longleftrightarrow V_w \otimes p_w V_{\mathcal{H}}^n$$

that intertwine the action of $q_w C_n$ on $q_w V_{\mathcal{H}}^n$ and the action of $\mathrm{End}(V_w) \otimes p_w C_n p_w$ on $V_w \otimes p_w V_{\mathcal{H}}^n$. Thus, a subspace $M \subset q_w V_{\mathcal{H}}^n$ is an irreducible C_n-module iff $p_w M$ is an irreducible $p_w C_n p_w$-module. The dimension formula also follows. But it is easy to see that $p_w C_n p_w = p_w(\mathbb{C}G)p_w = \mathbb{C}G_w$. Therefore, the isomorphism class of $p_w M$ as a $p_w C_n p_w$-module is determined by its isomorphism class as a G_w-representation. So the map $M \mapsto p_w M$ is an injection from $\mathcal{M}_{w,n}$ to $\Lambda_{w,n}$. By the definition of $\mathcal{M}_{w,n}$ and $\Lambda_{w,n}$, the map is obviously surjective. $\square$

Recall the natural isomorphism $V^n \cong (V^1)^{\otimes n}$. Using the isomorphisms $\phi : V^n \to V_{\mathcal{H}}^n$, we obtain an isomorphism

$$(8.4) \qquad\qquad\qquad i : V_{\mathcal{H}}^n \to (V_{\mathcal{H}}^1)^{\otimes n}.$$

By abuse of notation we also denote any isomorphism of the form $V_{\mathcal{H}}^{n+m} \cong V_{\mathcal{H}}^n \otimes V_{\mathcal{H}}^m$ that arises from ϕ by i.

Consider the Bratteli diagram of $B_n \subset B_{n+1}$. By (8.3) we may identify its vertices with $\mathcal{M}_n \cup \mathcal{M}_{n+1}$. Note that if $M \in \mathcal{M}_n$ and $M' \in \mathcal{M}_{n+1}$, the number of edges between them is $\dim(\mathrm{Hom}(M', i^{-1}(M \otimes V_{\mathcal{H}}^1)))$.

Lemma 8.6. *Fix n, a nonnegative integer, and $v \in O$. Take any $\lambda_v \in \Lambda_{v,n}$. Let $M \in \mathcal{M}_{v,n}$ be the irreducible C_n-module corresponding to λ_v via the bijection in Lemma 8.5. Fix $w \in O$, and let S be as in Definition 8.1. Then there is an isomorphism of G_w-representations*

$$(8.5) \qquad p_w^{n+1}(i^{-1}(M \otimes V_{\mathcal{H}}^1)) \cong \bigoplus_{u \in S} \mathrm{Ind}_{G_u \cap G_w}^{G_w} \left(\left(\mathrm{Res}_{G_u \cap G_w}^{G_u} \hat{g_u}(\lambda_v) \right) \otimes E_{u,w} \right).$$

Proof. By definition,

$$(8.6) \qquad\qquad p_w^{n+1}(i^{-1}(M \otimes V_{\mathcal{H}}^1)) = \bigoplus_{u \in o(v)} (p_u^n M) \otimes E_{u,w}$$

$$(8.7) \qquad\qquad\qquad\qquad = \bigoplus_{u \in S} G_w((p_u^n M) \otimes E_{u,w})$$

as vector spaces. As G_w-representations,

$$(8.8) \qquad G_w((p_u^n M) \otimes E_{u,w}) \cong \mathrm{Ind}_{G_u \cap G_w}^{G_w} \left(\left(\mathrm{Res}_{G_u \cap G_w}^{G_u} p_u^n M \right) \otimes E_{u,w} \right)$$

because $G_u \cap G_w$ is the stabilizer subgroup of $(p_u^n M) \otimes E_{u,w}$ in the left-hand side of (8.8). Finally,

$$(p_u^n M) = \hat{g_u}(p_v^n M) \cong \hat{g_u}(\lambda_v);$$

so the lemma follows from (8.7) and (8.8).   □

Lemma 8.7. *Let $M \in \mathcal{M}_n$ and $M' = E_{n+1}(i^{-1}(M \otimes V_{\mathcal{H}}^2))$, where E_{n+1} is the Jones projection. Then $M' \in \mathcal{M}_{n+2}$ and $M \leftrightarrow \lambda \implies M' \leftrightarrow \lambda$, where $\leftrightarrow$ denotes the bijection in Lemma 8.5.*

Proof. Let $M \in \mathcal{M}_{v,n}$, $v \in O$. Let S be the set of rooted paths of length two on $\mathcal{H}$ that have the form

$$*eve*$$

for some edge e and vertex v. Note that G_* fixes $\sum_{\rho \in S} \rho$. Then

$$p_v^{n+2} E_{n+1}(i^{-1}(M \otimes V_{\mathcal{H}}^2)) = E_{n+1}(i^{-1}((p_v^n M) \otimes V_{\mathcal{H}}^2))$$
$$= i^{-1}((p_v^n M) \otimes E_1(V_{\mathcal{H}}^2))$$
$$= i^{-1}\left(p_v^n M \otimes \mathbb{C}\left(\sum_{\rho \in S} \rho \right) \right),$$

which is equivalent to $p_v^n M \otimes_{G_v} 1$, as G_v-representations. Therefore, $p_v^n M \cong p_v^{n+2} M'$ as G_v-representations, which proves the result.   □

Proof of Theorem 8.2. By Theorem 7.1, we can calculate the principal graph of $R_U \subset R$ from the tower $(B_n)_n$. By Lemma 8.5 and the discussion preceding it, we have an injection for each n from the vertices of the Bratteli diagram of $B_n \subset B_{n+1}$ to the vertex set of Γ^0. As previously discussed, we identify the vertex set of this Bratteli diagram with $\mathcal{M}_n \cup \mathcal{M}_{n+1}$. $\mathcal{M}_n$ maps onto $\amalg_{o \in O} \Lambda_{o,n}$, and Lemma 8.6 implies that $\mathcal{M}_{n+1}$ maps onto all the vertices of Γ^0 that neighbor at least one vertex in $\amalg_{o \in O} \Lambda_{o,n}$. It also implies that this injection preserves the multiplicity of edges between pairs of vertices. Therefore, the injection extends to an isomorphism of the Bratteli diagram of $B_n \subset B_{n+1}$ with the subgraph of Γ^0 connected to $\amalg_{o \in O} \Lambda_{o,n}$. Note that the sole element of $\mathcal{M}_0$ is $V_{\mathcal{H}}^0$, which corresponds to the trivial G_*-representation. It follows that Γ is the principal graph of $R_U \subset R$ and 1_* is the distinguished vertex. If $M \in \mathcal{M}_{n,w}$, then the trace weight on M in the Bratteli diagram of $B_n \subset B_{n+1}$ is $\dim(M)$ divided by the normalization factor of k^n. Hence, the trace weight on M in Γ is $\dim(M) = c \cdot \dim(p_w^n M)$, where c is the order of $o(w)$ (Lemma 8.5). Finally, Lemma 8.7 ensures that the injection from simple summands of B_n to simple summands of B_{n+2} implemented by the Jones projection (the reflection property of the Bratteli diagram) is consistent with the injection induced by the graph Γ.   □

Corollary 8.8. *The subfactor $R_U \subset R$ has infinite depth iff its graph $\mathcal{H}$ is infinite.*

Proof. If $\mathcal{H}$ is finite, then O is a finite set and for each $v \in O$, G_v is a finite group, hence $\hat{G}_v$ is finite. So Γ is finite.

On the other hand, assume $\mathcal{H}$ is infinite. Then we claim that for every non-negative integer n there exists a vertex whose minimum distance from the root of $\mathcal{H}$ is n. For since every vertex of $\mathcal{H}$ is adjacent to exactly k edges (Remark 5.9),

there cannot be more than k^n vertices whose distance from the root is n. Fix any nonnegative integer n and let

$$\rho = *e_1 v_1 e_2 v_2 \ldots e_n v_n$$

be a path of minimal length from $*$ to v_n. Let $v_0 = *$. Then for each i, $0 \leq i \leq n$, the minimal distance from v_i to $*$ is i. Since distance to the root is preserved by the action of G, the vertices v_i are in pairwise distinct G-orbits. So we may choose O such that $v_i \in O$, $0 \leq i \leq n$. Then the trivial representation of G_{v_i}, denoted 1_i, is a vertex of Γ^0 for $0 \leq i \leq n$. Moreover, for each i, $0 \leq i \leq n - 1$, the vertices 1_i and 1_{i+1} are neighbors in Γ^0. Therefore, 1_i is a vertex of Γ for $0 \leq i \leq n$; hence Γ contains at least n vertices. Since n was arbitrary, Γ must be infinite.   □

9. Tree symmetry in Krishnan-Sunder subfactors

Let FG be the free group on the alphabet $[k]$ modulo the relation $a^{-1} = a$ for all $a \in [k]$. Consider the natural map $\mathcal{F} \to FG$. Given a word $w \in \mathcal{F}$, define its reduced form w_{red} to be the shortest word in $\mathcal{F}$ with the same image in FG as w. We may obtain w_{red} from w by repeatedly canceling double letters in w.

We now construct a graph $\mathcal{T}_k = (V_\mathcal{T}, E_\mathcal{T})$. Let $V_\mathcal{T}$ be the set of reduced words in $\mathcal{F}$ and let

$$E_\mathcal{T} = \{(u, v, a) : \ u, v \in V_\mathcal{T}, \ a \in [k], \text{ and } ua = v\}.$$

Define $\mathcal{T}_k$ to be the rooted graph $(V_\mathcal{T}, E_\mathcal{T})$, in which the endpoints of (u, v, a) are u and v, and the root vertex is the empty word. It is evident that $\mathcal{T}_k$ is a tree. (Let $a_1 \ldots a_n$ be any word of length n on $[k]$ without repetitions. Then $a_1 \ldots a_n$, as a vertex of $\mathcal{T}_k$, is in the nth generation of $\mathcal{T}_k$ and its parent vertex is $a_1 \ldots a_{n-1}$.) See Figure 2 for a sketch of the tree $\mathcal{T}_k$ in which each edge (u, v, a) is labeled by a. Clearly each vertex of $\mathcal{T}_k$ is the endpoint of exactly k edges labeled $1, 2, \ldots, k$.

Define the graph morphism $\pi : \mathcal{T}_k \to \mathcal{H}$ as follows:

$$\pi : \mathcal{T}_k \to \mathcal{H}$$
$$u \in V \mapsto [u] \in V_\mathcal{H}$$
$$(u, v, a) \in E \mapsto (\{[u], [v]\}, a) \in E_\mathcal{H}.$$

It is easy to check that π is well-defined. We will show that π is the universal covering map of $\mathcal{H}$ and gives rise to "tree symmetries" of Krishnan-Sunder subfactors.

We construct a tower of *tree algebras* from $\mathcal{T}_k$ in a fashion analogous to the construction of $(B_n)_n$. Let

$$\mathbb{C} = Q_0 \subset Q_1 \subset \ldots Q_n \subset \ldots$$

be the sequence of path algebras on the graph $\mathcal{T}_k$ (see the beginning of Section 7) with the trace given by a constant weight vector on $\mathcal{T}_k$. Let $V_\mathcal{T}^n$ denote the space on which Q_n acts, that is, the complex vector space with orthonormal basis the set of rooted paths of length n on $\mathcal{T}_k$. Denote the group of root-preserving automorphisms of $\mathcal{T}_k$ by $\mathrm{Aut}(\mathcal{T}_k)$. There is a natural action of $\mathrm{Aut}(\mathcal{T}_k)$ on the set of rooted paths of length n, and hence on $V_\mathcal{T}^n$. For each n let $A_n(k) \subset Q_n$ be the subalgebra commuting with the action of $\mathrm{Aut}(\mathcal{T}_k)$. Then $(A_n(k))_n$ is an increasing sequence of algebras with a faithful trace inherited from $(Q_n)_n$. Note that $(A_n(k))_n$ depends only on k, not on the biunitary U.
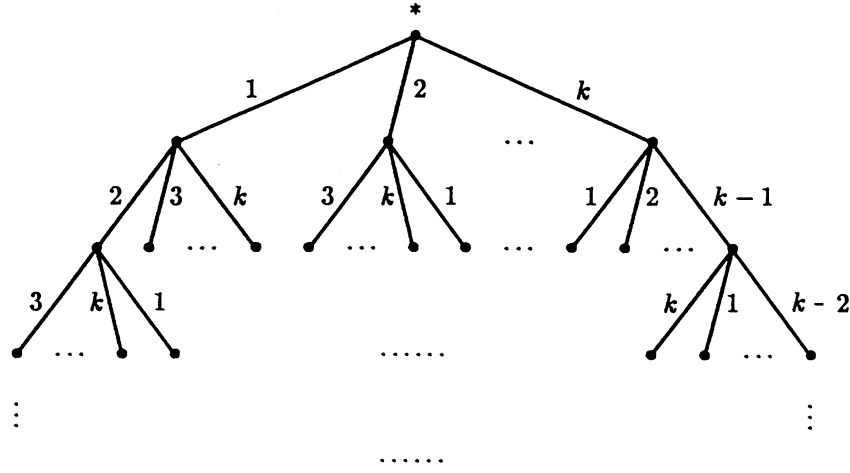
FIGURE 2. $\mathcal{T}_k$

Theorem 9.1. *Let $U \in M_p(\mathbb{C}) \otimes M_k(\mathbb{C})$ be a permutation biunitary and let $(B_n)_n$ be as defined in Section 7. There is a canonical injective $*$-morphism of algebras $\iota : \bigcup_n A_n(k) \to \bigcup_n B_n$ such that $\iota(A_n(k)) \subset B_n$ for each n and ι preserves the trace and the Jones projections.*

The proof appears after Corollary 9.5.

Proposition 9.2.
$$\pi^r : \mathcal{T}_k^r \to \mathcal{H}^r$$
is the universal covering map of $\mathcal{H}^r$ (π^r is defined in Section 4).

Proof. We will prove that π is a local isomorphism. The proof then follows from Lemma 4.1 (π is obviously surjective) and the fact that any tree is contractible (see, for example, [11]).

If $u \in V_{\mathcal{T}}$, then the set of edges in $\mathcal{T}$ which have u as an endpoint is $E_{\mathcal{T}}(u) = \{(u, ua, a) : a \in [k]\}$. The set of edges in $\mathcal{H}$ which have $[u]$ as an endpoint is
$$E_{\mathcal{H}}([u]) = \{(\{[u], [ua]\}, a) : a \in [k]\}.$$
Hence,

(9.1) $$\pi^r|_{E_{\mathcal{T}}(u)} : E_{\mathcal{T}}(u) \to E_{\mathcal{H}}([u])$$

is a bijection for each u. $\qquad\square$

Corollary 9.3. *The map π_*, induced by π, from the rooted paths of length n on $\mathcal{T}_k$ to the rooted paths of length n on $\mathcal{H}$ is a bijection for each n.*

Proof. This follows from the theory of covering maps (see, for example, [11]). $\quad\square$

Remark 9.4. Let $w = a_1 a_2 \ldots a_n$ be a word in $\mathcal{F}$ and let $\phi(w)$ be the rooted path on $\mathcal{H}$ associated to w (Section 7). Then
$$\pi_*^{-1}(\phi(w)) = v_0 e_1 v_1 e_2 \ldots e_n v_n$$
where v_0 is the empty word and $e_i = (v_{i-1}, v_i, a_i)$. We leave it to the reader to check that v_i is the reduced form of $a_1 a_2 \ldots a_i$ for each $i > 1$.

Corollary 9.5. *There is an injective group homomorphism*

$$\pi^* : \mathrm{Aut}(\mathcal{H}) \to \mathrm{Aut}(\mathcal{T}_k)$$

such that for all $g \in \mathrm{Aut}(\mathcal{H})$, $\pi\pi^*(g) = g\pi \in \mathrm{Hom}(\mathcal{T}_k, \mathcal{H})$.

Proof. This follows from the theory of covering maps (see, for example, [11]). $\square$

Let π_* also denote the induced unitary isomorphism of vector spaces:

$$\pi_* : V_{\mathcal{T}}^n \to V_{\mathcal{H}}^n.$$

Proof of Theorem 9.1. Let $A_n = A_n(k)$ for each n. For each n, let $\iota : \mathrm{End}(V_{\mathcal{T}}^n) \to \mathrm{End}(V_{\mathcal{H}}^n)$ be the injective $*$-morphism of algebras $\iota : F \mapsto \pi_* F \pi_*^{-1}$. It is obvious that $\iota(Q_n) \subset P_n$. Since π preserves the trace weights, ι is trace-preserving. By Corollary 9.5, if $F \in Q_n$ commutes with $\mathrm{Aut}(\mathcal{T}_k)$, then $\iota(F)$ commutes with $\mathrm{Aut}(\mathcal{H})$ and hence with the G-action. Therefore, $\iota(A_n) \subset B_n$ for each n. It is obvious that ι is compatible with the inclusions $A_n \subset A_{n+1}$ and $B_n \subset B_{n+1}$. So we have an injective trace-preserving morphism of filtered algebras $\iota : \bigcup_n A_n \to \bigcup_n B_n$.

It remains to show that ι preserves the Jones projections. If ρ is a rooted path of length n on a graph, let $\rho(i)$ denote the ith edge of ρ, $1 \le i \le n$. To show that π_* preserves Jones projections, it suffices to show that if ρ is a rooted path of length n on $\mathcal{T}$ and $\pi_*(\rho) = \mu$, then $\rho(n-1) = \rho(n) \iff \mu(n-1) = \mu(n)$. Direction $\implies$ is obvious. Assume $\mu(n-1) = \mu(n)$. Since $\mathcal{T}_k$ is a tree, $\rho(n-1) = \rho(n)$ iff the $(n-2)$nd and nth vertices of ρ are the same. Let

$$\mu = v_0 e_1 v_1 e_2 \ldots e_{n-1} v_{n-1} e_{n-1} v_{n-2}$$

and let

$$\mu' = v_0 e_1 v_1 e_2 \ldots e_{n-2} v_{n-2}$$

be the truncation of μ by its last two edges. Clearly μ^r on $\mathcal{H}^r$ is homotopic to $(\mu')^r$. It follows that the lifts of μ and μ' to $\mathcal{T}_k$ must end at the same vertex, since π is a covering map (Proposition 9.2). Let ρ' be the truncation of ρ by its last two edges. The lift of μ is ρ and the lift of μ' is ρ'; hence ρ' and ρ end at the same vertex. Therefore, the $(n-2)$nd and nth vertices of ρ are the same. $\square$

Proposition 9.6. *The inclusion ι in Theorem 9.1 is not surjective.*

Proof. Let $A_n = A_n(k)$ for each n. Let $\mathcal{T}_k^n$ be the subtree of $\mathcal{T}_k$ whose vertex set is those vertices whose distance from the root is at most n. Clearly $\mathcal{T}_k^n$ is stabilized by $\pi^*(G)$ and $\mathrm{Aut}(\mathcal{T}_k)$. Fix $n > p$. Let $G(n) \subset \mathrm{Aut}(\mathcal{T}_k^n)$ be the subgroup generated by the action of $\pi^*(G)$ on $\mathcal{T}_k^n$. We claim that it suffices to prove that $G(n)$ is a proper subgroup of $\mathrm{Aut}(\mathcal{T}_k^n)$. Assume this is so. Choose N such that there exists a rooted path ρ on $\mathcal{T}_k^n$ of length N that ends at the root and covers $\mathcal{T}_k^n$. Let $F \in \mathrm{End}(V_{\mathcal{T}}^N)$ be the orthogonal projection to $\mathrm{Span}(\pi^*(G)\rho)$, the span of the $\pi^*(G)$ orbit of ρ. It is easy to check that $\pi_* F \pi_*^{-1} \in B_N$. By construction of ρ, the dimension of $(\mathrm{Span}(\pi^*(G)\rho))$ is the order of $G(n)$. We have

$$\mathrm{Aut}(\mathcal{T}_k)(\pi^*(G)\rho) = \mathrm{Aut}(\mathcal{T}_k)\rho = \mathrm{Aut}(\mathcal{T}_k^n)\rho.$$

Again by construction of ρ, the dimension of $\mathrm{Span}(\mathrm{Aut}(\mathcal{T}_k^n)\rho)$ is the order of $\mathrm{Aut}(\mathcal{T}_k^n)$, which is greater than the order of $G(n)$ by assumption. Therefore, $\mathrm{Span}(\pi^*(G)\rho)$ is not stabilized by the action of $\mathrm{Aut}(\mathcal{T}_k)$, hence F is not contained in A_N. Therefore, $\pi_* F \pi_*^{-1}$ is not in $\iota(A_N)$. Since $A_{N+m} \cap (\mathrm{End}(V_{\mathcal{T}}^N) \otimes 1^m) = A_N \otimes 1^m$ for each positive integer m, $\pi_* F \pi_*^{-1}$ is not in the image of ι.

So it suffices to show that $G(n)$ is a proper subgroup of $\mathrm{Aut}(\mathcal{T}_k^n)$. This is easy, because $G(n)$ can be generated by p generators (by definition of G), but $\mathrm{Aut}(\mathcal{T}_k^n)$ requires n generators. (Here is a simple proof: if $\sigma \in \mathrm{Aut}(\mathcal{T}_k^n)$ let σ_i be the permutation given by the action of σ on the set of vertices that are distance i from the root. Let ϵ_i be the sign of σ_i. Then the map $\sigma \mapsto (\epsilon_i)_{1 \le i \le n}$ is a surjective group homomorphism onto $(\mathbb{Z}/2\mathbb{Z})^n$.) $\square$

10. THE INDEX 4 CASE

The subfactor $R_U \subset R$ has index 4 when $k = 2$. It will be helpful to keep the following examples in mind. Let $[k] = \{x_1, x_2\}$. Denote elements of $S_2 = \{(1), (12)\}$ with cycle notation, and let S_2 act on $[k]$ by its action on the subscripts of x_1 and x_2. Fix a positive integer n and choose p such that there is a permutation in S_{p-1} of order n. Define actions of $[k] = \{x_1, x_2\}$ and $[p] = \{\alpha_1, \ldots, \alpha_p\}$ on each other as follows. Let x_1 act identically on $[p]$. Let x_2 fix α_p and act on $\{\alpha_1, \ldots, \alpha_{p-1}\}$ with order n. Let $\alpha_1, \ldots, \alpha_{p-1}$ act identically on $[k]$. We consider two cases.

Suppose α_p acts identically on $[k]$. It is easy to see that these actions arise from a permutation biunitary U. Namely, U is given by

$$U_{\alpha a}^{\beta b} = \delta_{\alpha, b(\beta)} \cdot \delta_{a, \beta(b)}$$

for $\alpha, \beta \in [p]$ and $a, b \in [k]$. Denote this biunitary by $U(n, (1))$. In block form,

$$(10.1) \qquad\qquad U(n, (1)) = \begin{bmatrix} I_p & 0 \\ 0 & x_2 \end{bmatrix}$$

where I_p denotes the $p \times p$ identity matrix and x_2 here denotes the $p \times p$ permutation matrix for the x_2-action on row vectors.

In the second case, α_p acts as (12) on $[k]$. Again there is a permutation biunitary, which we denote $U(n, (12))$, that gives rise to these actions. Let x_2' denote the $(p-1) \times (p-1)$ permutation matrix, acting on row vectors, of the restriction of the x_2 action to $\{\alpha_1, \ldots, \alpha_{p-1}\}$. Then

$$(10.2) \qquad\qquad U(n, (12)) = \begin{bmatrix} (I_{p-1}, 0) & (0_{p-1}, 1) \\ (0_{p-1}, 1) & (x_2', 0) \end{bmatrix}$$

where $U(n, (12))$ is written in block form and each block entry (J, K) is a $p \times p$ block diagonal matrix with one $(p-1) \times (p-1)$ block J and one 1×1 block K.

We will show that when $U = U(n, (1))$, we have $(\mathcal{H}, G) = (A_{2n-1}^{(1)}, 1)$, and the principal graph is $A_{2n-1}^{(1)}$; and when $U = U(n, (12))$, we have $(\mathcal{H}, G) = (A_{2n-1}^{(1)}, S_2)$, and the principal graph is $A_1^{(2)}$, if $n = 1$, and $D_{n+2}^{(1)}$, if $n \ge 1$. Moreover, we will show this is a complete set of principal graphs of Krishnan-Sunder subfactors of index 4.

For the moment, fix any Krishnan-Sunder subfactor of index 4 arising from a permutation biunitary U. For $g \in S_2$, define

$$[p]_g = \{\alpha \in [p] : \ \alpha \text{ acts on } \{x_1, x_2\} \text{ like } g\}.$$

By Corollary 13 of [10], we may assume that

$$(10.3) \qquad\qquad x_1 \text{ acts identically on } [p],$$

$$(10.4) \qquad\qquad [p]_{(1)} \text{ is nonempty.}$$

Lemma 10.1.   (1) *The action of x_2 on $[p]$ stabilizes $[p]_{(1)}$ and $[p]_{(12)}$.*
(2) *If $g \in S_2$ and $\alpha \in [p]_g$, then α acts on $\mathcal{F}$ as the diagonal action of g.*

Proof. To prove part 1, it suffices to prove x_2 stabilizes $[p]_{(1)}$. Suppose there exists $\alpha \in [p]_{(1)}$ such that $x_2(\alpha) \in [p]_{(12)}$. Let $x_2(\alpha) = \beta$. Since $\alpha(x_2) = x_2$ and $x_2(\alpha) = \beta$, $U_{\beta x_2}^{\alpha x_2} = 1$. Also, $\beta(x_1) = x_2$ and $x_1(\beta) = \beta$; hence $U_{\beta x_2}^{\beta x_1} = 1$. But then U has two 1's in its (β, x_2) column. Contradiction.

We now prove part 2. Let $w = a_1 a_2 \ldots a_n \in \{x_1, x_2\}^n$ and $g \in S_2$. By part 1 and (3.1),

$$a_1 a_2 \ldots a_r([p]_g) \subset [p]_g$$

for each r, $1 \leq r \leq n$. Then by (3.3),

$$\alpha(a_1 a_2 \ldots a_n) = g(a_1)g(a_2)\ldots g(a_n).$$

$\square$

Proposition 10.2. *Let G and $\mathcal{H}$ be as defined in Sections 3 and 5. Then:*

(1) *$\mathcal{H}$ is the Coxeter graph $A_{2n-1}^{(1)}$, where n is the order of the action of x_2 on $[p]$.*

(2) *G is trivial if $[p]_{(12)}$ is empty, and $G \cong S_2$ otherwise.*

Proof. 1. By Remark 5.9, $\mathcal{H}$ is a bipartite graph in which every vertex is adjacent to exactly two edges. Evidently, $\mathcal{H}$ must be $A_{\infty,\infty}$ or $A_{2n-1}^{(1)}$ for some n. Let n be the smallest integer such that $(x_2 x_1)^n \sim e$, if such an n exists. By construction of $\mathcal{H}$, if n exists then $\mathcal{H} \cong A_{2n-1}^{(1)}$. Otherwise, $\mathcal{H} \cong A_{\infty,\infty}$.

We must show that n is the order of the x_2 action (which of course exists). Let m be any positive integer. By (3.1), (3.3), and (10.3),

(10.5)	$\alpha((x_2 x_1)^m) = (x_2 x_1)^m$	if $\alpha \in [p]_{(1)}$
(10.6)	$\alpha((x_2 x_1)^m) = (x_1 x_2)^m$	if $\alpha \in [p]_{(12)}$
(10.7)	$(x_2 x_1)^m(\alpha) = x_2^m(\alpha)$	for all $\alpha \in [p]$
(10.8)	$(x_1 x_2)^m(\alpha) = x_2^{-m}(\alpha)$	for all $\alpha \in [p]$

Note that the G-orbit of $(x_2 x_1)^m$ is a nonempty subset of

$$\{(x_2 x_1)^m, (x_1 x_2)^m\}.$$

So by (10.7), (10.8), and Remark 3.7, $(x_2 x_1)^m \sim e$ iff x_2^m acts identically on $[p]$. Therefore, n is the order of the x_2-action.

2. Immediate from Lemma 10.1 and (10.4). $\square$

Proposition 10.3. *The pairs $(\mathcal{H}, G)$ arising from permutation biunitaries U with $k = 2$ are exactly $(A_{2n-1}^{(1)}, 1)$ and $(A_{2n-1}^{(1)}, S_2)$, $n = 1, 2, 3, \ldots$. In particular, if $U = U(n, (1))$, then $(\mathcal{H}, G) = (A_{2n-1}^{(1)}, 1)$, and if $U = U(n, (12))$, then $(\mathcal{H}, G) = (A_{2n-1}^{(1)}, S_2)$.*

Proof. By Proposition 10.2, the pairs listed above are the only possible pairs $(\mathcal{H}, G)$. So it suffices to prove the second statement, which follows directly from Proposition 10.2. $\square$

Corollary 10.4. *The principal graphs arising from Krishnan-Sunder subfactors of index 4 are $A_{2n-1}^{(1)}$, $n \geq 1$, and $D_n^{(1)}$, $n \geq 4$. In particular, if $U = U(n, (1))$, then the principal graph of $R_U \subset R$ is $A_{2n-1}^{(1)}$; if $U = U(1, (12))$, then the principal graph of $R_U \subset R$ is $A_1^{(1)}$; and if $U = U(n, (12))$, with $n \geq 2$, then the principal graph of $R_U \subset R$ is $D_{n+2}^{(1)}$.*

Proof. We need only calculate Γ, using Theorem 8.2, from the pairs $(\mathcal{H}, G)$ listed in the previous proposition. The cases $(\mathcal{H}, G) = (A_{2n-1}^{(1)}, 1)$ and $(\mathcal{H}, G) = (A_1^{(1)}, S_2)$ are both extreme cases, discussed by Krishnan and Sunder ([10], see the end of Section 5). The corresponding principal graphs are $A_{2n-1}^{(1)}$ and $A_1^{(1)}$, respectively (see Remark 8.4). If $(\mathcal{H}, G) = (A_{2n-1}^{(1)}, S_2)$, $n \geq 2$, then Remark 8.3 applies. It is easy to verify that $\Gamma = D_{n+2}^{(1)}$. Let $\{* = v_0, v_1, \ldots, v_{2n-1}\}$ be the vertex set of $A_{2n-1}^{(1)}$, $\{e_1, e_2, \ldots, e_{2n}\}$ the edge set, and $\{v_{i-1}, v_i\}$ the endpoint set of the edge e_i, $0 \leq i \leq 2n$, taking indices of vertices mod $2n$. Then the action of $(12) \in G$ on $A_{2n-1}^{(1)}$ is given by the vertex map $v_i \mapsto v_{2n-i}$. Using the notation of Theorem 8.2, we may set $O = \{v_0, v_1, \ldots, v_n\}$. Given $v \in O$, if $v = v_0$ or $v = v_n$, then $G_v = S_2$. Otherwise G_v is trivial. Thus, v_0 and v_n each "split" into two vertices in Γ. Using the simple induction rules for S_1 and S_2, we obtain the graph $D_{n+2}^{(1)}$. $\square$

11. AN INFINITE-DEPTH KRISHNAN-SUNDER SUBFACTOR

Set $k = p = 3$. Let $[k] = \{1, 2, 3\}$ and $[p] = \{\alpha, \beta, \gamma\}$. Consider the following bi-unitary matrix U in $M_p(\mathbb{C}) \otimes M_k(\mathbb{C})$, constructed by Krishnan and Sunder (example (15) in [10]):

$$U \;=\; \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

We have written U in block form, where blocks are indexed, in order, by α, β, and γ, and each block is indexed, in order, by 1, 2, and 3.

Krishnan and Sunder showed that $R_U \subset R$ is an irreducible subfactor of infinite depth, and characterized the principal graph in terms of finite dimensional representations of subgroups of G. We will obtain a simple proof of their result, based on our general theory, and show that the principal graph is not a tree.

The biunitary U gives rise to the following actions of $[k]$ and $[p]$ on one another:

$$
\begin{array}{lll}
\alpha : 1 \mapsto 1 & \beta : 1 \mapsto 2 & \gamma : 1 \mapsto 3 \\
\quad 2 \mapsto 2 & \quad 2 \mapsto 1 & \quad 2 \mapsto 2 \\
\quad 3 \mapsto 3 & \quad 3 \mapsto 3 & \quad 3 \mapsto 1 \\
\\
1 : \alpha \mapsto \alpha & 2 : \alpha \mapsto \alpha & 3 : \alpha \mapsto \beta \\
\quad \beta \mapsto \beta & \quad \beta \mapsto \beta & \quad \beta \mapsto \alpha \\
\quad \gamma \mapsto \gamma & \quad \gamma \mapsto \gamma & \quad \gamma \mapsto \gamma
\end{array}
$$

Lemma 11.1. *For all integers $n \geq 0$, $\pi^*(G)$ acts transitively on the set of vertices in the nth generation of $\mathcal{T}_3$, where $\pi^* : \mathrm{Aut}(\mathcal{H}) \to \mathrm{Aut}(\mathcal{T}_3)$ is as in Corollary 9.5.*

Proof. The proof is by induction. Since only the root is in the 0th generation, the lemma is obvious for $n = 0$. Let I denote the set of vertices in the $(n-1)$st generation of $\mathcal{T}_3$. Assume that $n > 0$ and $\pi^*(G)$ acts transitively on I. We must prove that $\pi^*(G)$ acts transitively on the set of vertices in the nth generation of $\mathcal{T}_3$. For each vertex $v \in I$, v has exactly 2 children, which we denote v_1 and v_2. We claim that it suffices to find $v \in I$ and $g \in \pi^*(G)$ such that $g(v_1) = v_2$. For suppose we have such a v and g. Let $x, y \in I$. By the induction hypothesis there exist $h_x, h_y \in \pi^*(G)$ such that $h_x(x) = v$ and $h_y(y) = v$. Then either $h_y^{-1} h_x$ or $h_y^{-1} g h_x$ maps x_i to y_j for any $i, j \in \{1, 2\}$. This proves the claim.

We now exhibit such a v and g. Set

$$h = \gamma\alpha\beta\gamma\alpha\beta\gamma\alpha\beta,$$

$$g = \alpha h.$$

Each element of $[p]$ is its own inverse as a permutation of S_k; hence $g \in G$. Let v be the word of length $n-1$ of the form

$$v = (121212\ldots)3.$$

Since v is a reduced word of length $n-1$, v is a vertex in the $(n-1)$st generation of $\mathcal{T}_3$. The two children of v are

$$v_1 = (121212\ldots)31$$

and

$$v_2 = (121212\ldots)32.$$

We must show that $g(v_1) = v_2$.

We claim that h acts identically on $\mathcal{F}$. It is easy to check that $\alpha\beta$ acts on $\mathcal{F}$ as the diagonal action of the cycle $(12) \in S_3$ and γ acts on $\mathcal{F}$ as the diagonal action of the cycle $(13) \in S_3$. The claim follows since $(13)(12)(13)(12)(13)(12)$ equals the identity in S_3.

It is also easy to check that $\alpha(v_1) = v_2$. Therefore, $g(v_1) = \alpha h(v_1) = \alpha(v_1) = v_2$. So we are done. $\qquad\qquad\square$

Proposition 11.2. *The graph morphism* $\pi : \mathcal{T}_3 \to \mathcal{H}$, *as defined in Section 9, is an isomorphism.*

Proof. Since π is a covering map (Proposition 9.2), it suffices to show that the inverse image of the root vertex of $\mathcal{H}$ contains only the root vertex of $\mathcal{T}_3$. In other words, we must show that if w is a reduced word in $\mathcal{F}$ of positive length n, then $\pi(w)$ is not the root of $\mathcal{H}$. By Lemma 11.1 there exists $g \in G$ such that $g(w)$ is the word of length n of the form $w' = 3(121212\ldots)$. It is easy to check that $w'(\alpha) = \beta$. So w' does not act identically on $[p]$, hence w' does not act identically on $[p]^2$, hence $w' \not\sim e$, and hence $w \not\sim e$. Therefore, $\pi(w)$ is not the root of $\mathcal{H}$. $\qquad\square$

Remark 11.3. We can apply Theorem 8.2 to $(\mathcal{H}, G)$ to obtain the description of the principal graph given by Krishnan and Sunder in Theorem 37 of [10]. Let $* = v_0, v_1, v_2, v_3, \ldots$ be a sequence of vertices in $\mathcal{H} \cong \mathcal{T}_3$, such that v_n neighbors v_{n+1} and the distance from v_n to the root is n. By Lemma 11.1, $O = \{v_n\}_0^\infty$ is a set of representatives of the G-orbits in the vertex set of $\mathcal{H}$. Let $G_n = G_{v_n}$, the subgroup of G that fixes v_n, for each n. It is easy to verify that $(\mathcal{H}, G)$ satisfies the hypotheses of Remark 8.3. Let us apply Theorem 8.2. The vertex set of Γ^0 is $\mathrm{II}_n \hat{G}_n$. Clearly $G_{n+1} \subset G_n$ for each n. So if $\lambda_n \in \hat{G}_n$ and $\lambda_{n+1} \in \hat{G}_{n+1}$, then the

number of edges between λ_n and λ_{n+1} is $\dim(\mathrm{Hom}(\lambda_{n+1}, \mathrm{Res}^{G_n}_{G_{n+1}} \lambda_n))$. Finally, Γ is the connected component of Γ^0 containing 1_*.

We now show that Γ is not a tree, contrary to Krishnan and Sunder's expectations in [10]. Let $\mathcal{T}_3(2)$ be the subtree of $\mathcal{T}_3$ within radius 2 of the root. It is the tree with 10 vertices in which the root vertex has 3 children and each of those children has 2 children. For $n = 0, 1$, and 2, let $H_n \subset \mathrm{Aut}(\mathcal{T}_3(2))$ be the subgroup generated by the action of G_n on $\mathcal{T}_3$, restricted to $\mathcal{T}_3(2)$. Each H_n is a quotient of G_n, and there is a natural injection $\hat{H}_n \to \hat{G}_n$ that commutes with Res. Thus, if we replace $\mathcal{H}$ by $\mathcal{T}_3(2)$ and G_n by H_n in Definition 8.1, we obtain a subgraph of Γ^0, which we denote $\Gamma^0(2)$. Let $\Gamma(2)$ be the corresponding subgraph of Γ.

The vertices in the second generation of $\mathcal{T}_3(2)$ are 12, 13, 21, 23, 31, and 32. Number these 1 through 6 respectively, and consider the image of H_n in S_6, for $n = 0, 1$, and 2. It is easy to verify that $\alpha \mapsto (56)$, $\beta \mapsto (13)(24)$, and $\gamma \mapsto (16)(25)(34)$. A calculation using GAP ([1]) shows that $|H_0| \geq 48$. But, as will be obvious below, $\mathrm{Aut}(\mathcal{T}_3(2))$ has order 48, so $H_0 = \mathrm{Aut}(\mathcal{T}_3(2))$.

It is well known that the automorphism groups of trees such as $\mathcal{T}_3(2)$ are wreath products of permutation groups. We have

$$
\begin{array}{rclcl}
H_0 = \mathrm{Aut}(\mathcal{T}_3(2)) & \cong & S_3 \wr S_2 & = & S_3 \ltimes (S_2 \times S_2 \times S_2) \\
H_1 & \cong & (S_2 \wr S_2) \times S_2 & = & (S_2 \ltimes (S_2 \times S_2)) \times S_2 \\
H_2 & \cong & (S_2 \wr S_2) & = & S_2 \ltimes (S_2 \times S_2)
\end{array}
$$

with the inclusion maps

$$H_1 \to H_0$$
$$(\sigma(g_1, g_2), g_3) \mapsto \sigma(g_1, g_2, g_3)$$

where $g_i, \sigma \in S_2$ and $\sigma \in S_3$, via the usual inclusion $S_2 \subset S_3$;

$$H_2 \to H_1$$
$$\sigma(g_1, g_2) \mapsto (\sigma(g_1, g_2), (1))$$

where $\sigma, g_i \in S_2$ and (1) is the identity element of S_2.

The representation theory of these groups is well known ([5]). The irreducible representations of $S_n \wr S_m$ are in bijection with the set of objects given by: a choice of r distinct irreducible representations $\nu_i \in \hat{S}_m$, $1 \leq i \leq r$; positive integer weights n_i, $1 \leq i \leq r$, such that $\sum^r n_i = n$; and a choice of $\mu \in (\prod^r S_{n_i})\hat{}$. The object is defined up to permutations of the ν_i (with the corresponding permutation of weights, etc.). We denote the corresponding representation by $(\mu)(\bigotimes^r \nu_i^{n_i})$. The restriction rules are:

$$\mathrm{Res}^{H_1}_{H_2}\left((\mu)(\bigotimes \nu_i^{n_i}) \otimes \nu\right) = \dim(\nu) \cdot (\mu)(\bigotimes \nu_i^{n_i})$$

where $\mu, \nu_i \in \hat{S}_2$ and $\sum n_i = 2$;

$$\mathrm{Res}^{H_0}_{H_1}(\mu)(\otimes^r \nu_i^{n_i}) = \bigoplus_{j=1}^r (\mathrm{Res}^{\prod S_{n_i}}_{\prod S_{n_i'}} \mu)(\bigotimes^r \nu_i^{n_i'}) \otimes \nu_j$$

where $\mu \in \hat{S}_3$, $\nu_i \in \hat{S}_2$, $\sum^r n_i = 3$, and

$$n_i' = n_i - \delta_{i,j}.$$

Theorem 11.4. *The principal graph of $R_U \subset R$ is not a tree.*

Proof. By the previous discussion it suffices to show that $\Gamma(2)$ contains a cycle. Let 1_1 denote the trivial representation of the trivial group; let 1_2 and -1_2 denote the trivial and alternating representations of S_2; and let 1_3, ν_3, and -1_3 denote the trivial, standard, and alternating representations of S_3. Let λ_i, $0 \leq i \leq 9$, be the following elements, respectively:

$$(1_2 \otimes 1_1)(1_2^2 \otimes -1_2) \in \hat{H}_0$$
$$(1_1 \otimes 1_1)(1_2 \otimes -1_2) \otimes 1_2 \in \hat{H}_1$$
$$(-1_2 \otimes 1_1)(1_2^2 \otimes -1_2) \in \hat{H}_0$$
$$(-1_2)(1_2^2) \otimes -1_2 \in \hat{H}_1$$
$$(-1_2)(1_2^2) \in \hat{H}_2$$
$$(-1_2)(1_2^2) \otimes 1_2 \in \hat{H}_1$$
$$(\nu_2)(1_2^3) \in \hat{H}_0$$
$$(1_2)(1_2^2) \otimes 1_2 = 1_{H_1} \in \hat{H}_1$$
$$(1_2)(1_2^2) = 1_{H_2} \in \hat{H}_2$$
$$(1_2)(1_2^2) \otimes -1_2 \in \hat{H}_1$$

Then the λ_i, $0 \leq i \leq 9$, are vertices of $\Gamma^0(2)$, and it is straightforward to check that there is an edge between λ_i and λ_{i+1} for each i, taking indices modulo 10. Therefore, $\Gamma^0(2)$ contains a cycle. Finally, λ_7, the trivial representation of H_1, is clearly connected to the trivial representation of H_0, so $\Gamma(2)$ contains a cycle. $\quad\square$

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